19 OCT 2015 wk4 Foundations of Computer Science Jed Gibbs for Charlton Rodda

SUM (OR DISJOINT UNION)

distinguishes the two copies of the elements
- we tag elements of x with a zero and y with a 1
(see slide)

(x and y are finite)

$$|X \cup Y| \geq |X|$$
$$|X \cup Y| \geq |Y|$$
$$|X \cup Y| \leq |X| + |Y|$$

when will cardinality of $|X + Y| = |X \cup Y|$?
(when they share elements)

$$X + Y + Z \stackrel{\text{def}}{\equiv} \{(x, 0) \mid x \in Y\} \cup \{(y, 1) \mid y \in Y\} \cup \{(z, 2) \mid z \in Z\}$$

Functions as Sets of Ordered Pairs

In set theory a function is identified with its "graph" a set of ordered pairs.

The identity function on $\{0, 1\}$ is identified with the following set $\{(0,0), (1,1)\}$.

$f : X \to Y$ graph$(f) \subseteq X \times Y$

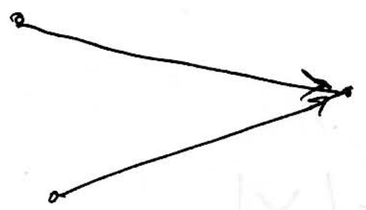
If for some $x \in X$, we have (x, y) and (x, y') elements of graph(f) then it must be true that $y = y'$

For all $x \in X$, there exists a $y \in Y$

such that $(x, y) \in \text{graph}(f)$

for each x there exists unique y such that

$(x, y) \in \text{graph}(f)$



A function x is $(1+1)$ if

if it satisfies $(+)$

whenever $f(x) = f(x')$

then $x = x'$

FUNCTION SPACES

· given sets X and Y, let Y^X denote the set of all functions from X to Y

if X and Y are finite sets, what is the cardinality of of Y^X?

Important function ev

How many functions are there from X to Y?

$|Y|^{|X|}$ ✓ $|Y^X| = |Y|^{|X|}$

let's count functions:

Suppose $|X| = m$, in particular

$$X = \{x_1, x_2, x_3 \ldots x_m\}$$

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$|Y| = n$ then $Y = \{y_1, y_2, y_3 \cdots y_n\}$

& n choices where to find x_1, x_2, x_m

FUNCTION SPACES

write down all the elements of the following function spaces:

2^0, 2^1 and 2^2

$0 = \emptyset$
$1 = \{0\}$
$2 = \{0, 1\}$
$\underline{n} = \{x \mid 0 \leq x < n\}$
$x \in \mathbb{N}$

2^0 no elements $= 1$

2^1 $\{(0, 0)$ $(0, 1)\}$

2^2 $\{(0, 0)$ $\{0, 1\}$ $\{1, 0\}$ $\{1, 1\}\}$

$2^0 = \, ! : \emptyset \to \underline{2}$

graph $(!) \subseteq \emptyset \times 2$
$= \emptyset$

graph$(!) = \emptyset$ 1 subset, itself $2^0 = \{\emptyset\}$

2^1 cardinality $= 2$ so 2 functions

$\underline{1} \to \underline{2}$
$\{0\} \to \{0, 1\}$ $\{\{(0,0)\}, \{(0,1)\}\}$

$$2^2 = \{\{(0,0),(1,1)\}$$
$$\{(0,1),(1,0)\}$$
$$\{(0,0),(1,0)\}, \{(0,1)(1,1)\}\}$$

Relations between sets

2 sets X and Y have cartesian product $X \times Y$ the set of ALL ordered pairs (x, y) with $x \in X$ and $y \in Y$.

A binary relation R

A relation from X to Y is a subset of the cartesian product $X \times Y$

Given a set X the powerset of X, written $P(X)$ is the set of all subsets of X

example $P(\phi) = \{\phi\}$
$P(\{0,1\})$
$= \{\phi, \{0\}, \{1\}, \{0,1\}\}$

If X is finite, $|P(X)| = 2^{|X|}$ (2 because each element may be present in the subset or not "<u>to be</u> or <u>not to be</u>")

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Example (DIGRAPHS)

A directed graph (digraph).

* There are two slides about digraphs left as homework.

COMPOSITION OF RELATIONS

if R is a relation from X to Y and S is a relation from Y to Z then R ; S is a relation from X to Z defined.

associative

Question - all people since 0 AD / CE

If X and Y are finite sets, how many different ~~relationships~~ relations are there from X to Y?

Let R be the relation on the set ~~of~~ of all people defined $(x, y) \in R$ when x is a parent of y
① what is $R ; R$? what is $R ; R ; R$?
X - all people
$R \subseteq X \times X$
$(y, x) \in R$ exactly when y is a parent of x
$R ; R$
$(z, x) \in P, R$ such that

INJECTIVE FUNCTIONS

function $f: X \to Y$ is injective (or 1-1, or into)
if it satisfies the following condition:

different elements go to different:

SURJECTIVE FUNCTIONS

function $f: X \to Y$ is surjective (or onto)

functive is bijective (or 1-1 correspondence) if
both injective and surjective).

Questions

① Are there any injective functions from a set with 10 elements to a set with 2? None

② How many functions are there from a set with 10 elements to a set with 2 / how many surjective? $\to 2^{10} = 1024$
$2^{10} - 2$ (the two missing are those where everything is mapped to 0 or 1)

* Homework
given any 2 natural numbers, ~~~~ how many surjective functions are there from a set size n to a set size m?

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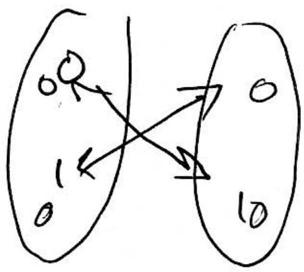
INVERTING FUNCTIONS

inverse of



is itself.

ISOMORPHISM THEORUM

Let $f : X \to Y$ be a function. Then the following are equivalent:

f is bijective (injective and surjective)

f has an inverse.

Proof ('bijective' implies 'has inverse')

If f is bijective then for ~~any~~ every element $y \in Y$ there is a unique element $x \in X$ such that $f(x) = y$ thus letting $f^{-1}(y) = x$ defines a function $f^{-1} : Y \to X$

(see slide for Proof continued)

'Bureacracy'

$$X \times (Y \times Z) \neq (X \times Y) \times Z$$

$$\cong$$

* TOMORROW Help session
* THURSDAY TEST